First Passage Time Interdiction in Markov Chains

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Abstract

We introduce a new variant of the network interdiction problem with a Markovian evader that randomly chooses a neighboring vertex in each step to build their path from designated source(s) to terminal(s). The interdictor's goal is to maximize the evader's minimum expected *first passage time*. We establish sufficient conditions for the interdiction to not be counter-productive, prove that the problem is NP-hard, and demonstrate the model's usefulness by solving a mixed-integer programming formulation on a test bed of social networks.

Keywords: Network interdiction, Markov influence graphs, Markovian evader interdiction

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1. Introduction

Consider a network in which a 'Markovian evader' moves from one vertex to another vertex along an arc. In this setting the vertices of the network form the state space of this Markov chain and transitions happen from a vertex to its (out) neighbor with known one-step transition probabilities. This stochastic model underlies many approaches analyzing misinformation and influence spread in social networks, cluster analysis of hyperlink networks found in the World Wide Web [1], and controlling infectious diseases where the Markov chain describes the disease spread [2]. In particular, this study is motivated by the use of Markov chains in the stochastic variants of attack graphs used in cybersecurity analysis [3, 4, 5, 6]. Specifically, a probabilistic attack graph can be modeled as a Markovian evader moving in the graph. Each state of the graph represents a vulnerability of the system (with one absorbing state representing the violated security goal); and the transition probabilities are a function of the exploitability scores of the vulnerabilities [3].

A relevant question from the point of view of a network manager is to allocate resources to maximally disrupt, or interdict, the evader's operation, and several variants of the Markovian evader interdiction framework are available in the literature on network interdiction. For example, Gutfraind et al. [7] consider multiple Markovian evaders who choose edges to traverse based on a random walk defined by a Markovian transition matrix. Each evader has a target in the network, and the goal of the interdictor is to interdict edges using a limited budget to increase the probability of capturing evaders before reaching their targets. Johnson et al. introduce two interdiction problems with Markovian evaders in [8], wherein one problem maximizes the number of captured evaders under a limited budget for vertex interdiction and the other problem focuses on capturing all the evaders at a minimum cost. Sefair et al. [9] consider a setting where the interdictor protects a subset of vertices with a limited budget while the evader attacks a set of unprotected vertices leading to changes in the transition probabilities. The evader's goal is to minimize the weighted expected hitting time, while the interdictor seeks to maximize it.

In this paper, we introduce a new variant of Markovian evader interdiction where the interdiction of a vertex increases the probability that the Markov chain remains in said vertex and decreases the probability that the Markov chain jumps to other vertices. As a consequence, interdiction 'slows down' the evolution of the chain. In particular, we consider the expected *first passage time* as the metric of performance of the evader, and the interdictor aims to maximize the minimum expected first passage time between two given sets of vertices. For instance, in the context of attack graph interdiction, our model could serve to identify the vulnerabilities where a network manager should invest his/her limited resources to reduce exploitability scores, in order to maximize the time it takes to attackers to reach their objective.

Previous papers have consider an approach to Markov chain interdiction analogous to the one we propose. For instance, Magazev and Tsyrulnik [10] and Kasenov et al. [11] consider interdiction to maximize the expected first passage time between two fixed nodes in the context of attack graph interdiction. These works, however, assume specific simple Markov chains and their results cannot be employed in general Markov chains. On the other hand, works that consider interdiction in general Markov chains to optimize first passage times, such as [9], focus on developing solution algorithms but provide no general properties about the model nor its computational complexity. In this sense, our work complements existing literature by providing a general model and theoretical analyses for first passage time interdiction in Markov Chains.

Specifically, the main contributions of this paper are: (*i*) we demonstrate that, under some circumstances, interdiction could *improve* the evader's performance contrary to expectation. We then provide sufficient conditions for designing an interdiction policy that prevents counter-productive interdiction; namely, that the interdiction penalties only depend on the departing state (Section 3.1); (*ii*) we show that the optimal interdiction problem is NP-hard by a reduction from vertex cover (Section 3.2); and (*iii*) we introduce a mixed-integer programming (MIP) formulation and use it to demonstrate the usefulness of this interdiction framework on a test bed of benchmark social and biological networks (Sections 4 and 5).

The remainder of the paper is organized as follows. Section 2 introduces the problem formally and Section 3 contains our theoretical results. Section 4 has the MIP formulations which are tested in the numerical experiments in Section 5. Section 6 presents our concluding remarks.

2. Problem statement

Consider a digraph $D = (V, A)$ with vertex set V and arc set $A \subseteq V \times V$ [12]. We assume that every vertex $i \in V$ has a self-loop $(i, i) \in A$, but D contains no parallel arcs. The *open* out-neighborhood of vertex i is defined as $N^+(i) \coloneqq \{j \in V \setminus \{i\} : (i, j) \in A\}$ and $N^+[i] \coloneqq N^+(i) \cup \{i\}$ is the *closed* out-neighborhood. We assume that the stochastic process $\{X_n : n = 0, 1, 2, \ldots\}$ is a discrete-time Markov chain (DTMC) with state space V. The transition probability matrix is denoted by P, where $P_{ij} = 0$ if $(i, j) \notin A$.

For $j \in V$ let τ_j be the first time by which the DTMC visits vertex j, which is defined as $\tau_j := \min\{n \geq 0 : X_n = j\}$. For $i \in V$, we define $t_{ij} := \mathrm{E}[\tau_j | X_0 = i]$, thus t_{ij} is the *expected first passage time* (FPT) to j given that the chain initially is at state i. For convenience, hereafter we omit the word 'expected' when referring to first passage times. Consider two disjoint, nonempty subsets of vertices denoted by S and T. In the context of misinformation spread, subset S can be considered as malicious users from which misinformation originates, whereas those in T can be considered as users vulnerable to misinformation. In attack graphs, S can be considered as a set of initial vulnerability states, whereas T is a set of states representing the completion of the attack. We assume that $P[\tau_j < \infty | X_0 = i] = 1$ for all $i \in S$ and $j \in T$, which, because $|V| < \infty$, can be ensured if there is at most one closed communicating class in $C \subseteq V$ such that $T \subseteq C$ [13]. Under this assumption, for $j \in T$, the FPT to j starting in $i \in S$ can be computed by solving the following system of linear equations (Theorem 3.3, [14]):

$$t_{ij} = 1 + P_{ii} t_{ij} + \sum_{k \in N^+(i)} P_{ik} t_{kj}, \qquad \forall i \in V \setminus \{j\}, \tag{1a}$$

$$t_{jj} = 0. \tag{1b}$$

The goal of the interdiction problem is to disrupt vertices in V to increase the first passage times from vertices in S to vertices in T. We assume that interdicting vertex $i \in V$ decreases the transition probability P_{ij} to $P_{ij}(1 - \Delta_{ij})$ for every $j \in N^+(i)$, where $\Delta_{ij} \in [0, 1)$ is a known interdiction penalty parameter associated with every outbound arc at vertex i. (Note that we do not consider the self-loop an outbound arc, and self-loops are not associated with any interdiction penalty.) Consequently, interdicting vertex i increases the probability of traversing the self-loop at i to $P_{ii} + \sum_{j \in N^+(i)} P_{ij}\Delta_{ij}$ to ensure that the total one-step transition probability at i equals one. This is illustrated in Figure 1. Furthermore, as $\Delta_{ij} < 1$, the DTMC has the same class decomposition pre- and post-interdiction.

The motivation behind the proposed model of interdiction is to 'slow down' the evolution of the chain in the following sense. Whenever the DTMC visits an interdicted vertex, it will remain at the interdicted vertex for more time periods (in expectation) because we have $P_{ii} \leq P_{ii} + \sum_{j \in N^+(i)} P_{ij}\Delta_{ij}$ for any admissible value of Δ_{ij}. Furthermore, the expected number of time periods that the DTMC remains at an interdicted vertex can be made arbitrarily large by making all the Δ_{ij} arbitrarily close to 1. However, as we elaborate in Section 3.1, interdiction might not necessarily increase the first passage times and conditions are needed to ensure this happens.

Based on this interdiction model, we introduce the optimization problem (2) with the objective of interdicting at most b vertices to maximize the smallest FPT from S to T. Let $B := \left\{x \in \{0,1\}^{|V|} : \sum_{i \in V} x_i \leq b\right\}$ denote the set of feasible interdiction policies. Then, we aim to solve:

$$t^*_{S,T} := \max_{x \in B} \left\{ \min_{\substack{i \in S \\ j \in T}} \{t_{ij}(x) : t_{ij}(x) \text{ satisfies equations (3)}\} \right\}, \text{ where,} \tag{2}$$

$$t_{ij}(x) = 1 + \left(P_{ii} + \sum_{k \in N^+(i)} P_{ik}\Delta_{ik}x_i\right) t_{ij}(x)$$

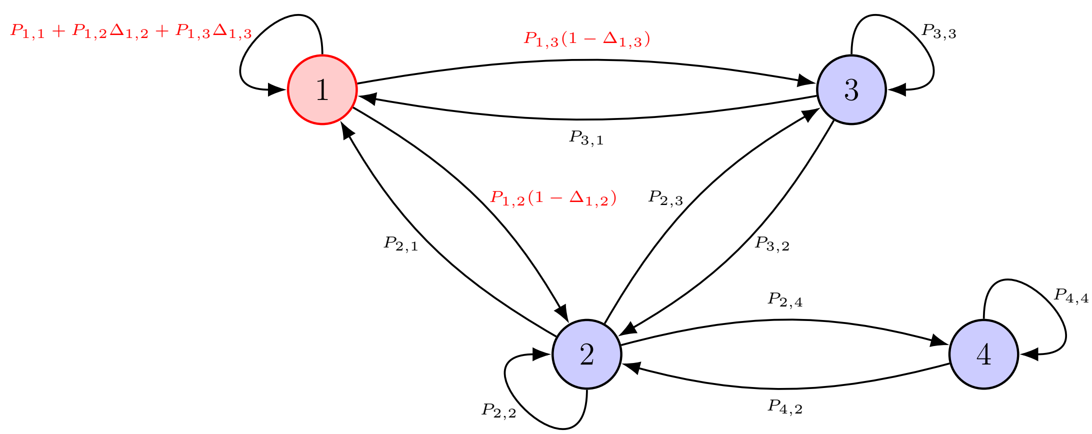

Figure 1: By interdicting vertex 1, self-loop transition probability at vertex 1 increases, while transition probabilities on outbound arcs $\{1,2\}$ and $\{1,3\}$ decrease.

$$+ \sum_{k \in N^+(i)} P_{ik}(1 - \Delta_{ik} x_i) t_{kj}(x) \qquad \forall (i,j) \in V \times V : i \neq j, \quad \text{(3a)}$$

$$t_{jj}(x) = 0 \qquad \forall j \in V. \quad \text{(3b)}$$

Observe that system of equations (3) is analogous to the system of equations in (1), with the difference that (3) captures the effect of interdiction according to policy $x \in B$.

3. Properties of the interdiction problem

In this section, we focus on the following: (i) establishing sufficient conditions under which interdicting a vertex does not decrease the FPTs; and *(ii)* proving that the optimization problem (2) is NP-hard.

3.1. Preventing counter-productive interdiction

In general, interdiction of a vertex might not ensure that the FPTs increase. Consider the digraph in Figure 2. Let $S = \{1\}$, $T = \{3\}$, and for every $i \in V$ let $P_{ij} = 1/3$ for each $j \in N^+[i]$. We can solve system (1) to find that $t_{13} = 4.5$. Indeed, note that the system of equations in this case is

$$\begin{aligned} t_{13} &= 1 + \frac{1}{3}t_{13} + \frac{1}{3}t_{23} \\ t_{23} &= 1 + \frac{1}{3}t_{23} + \frac{1}{3}t_{13} + \frac{1}{3}t_{43} \\ t_{43} &= 1 + \frac{1}{3}t_{43} + \frac{1}{3}t_{23}, \end{aligned}$$

and consequently, $t_{13} = 4.5$, $t_{23} = 6$ and $t_{43} = 4.5$.

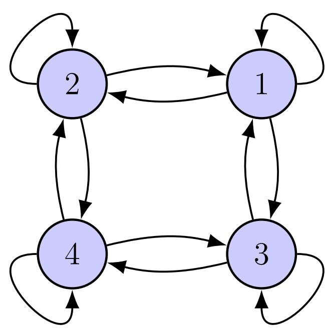

Figure 2: An example to illustrate that FPT may decrease upon interdiction in some cases.

Now, suppose that $\Delta_{12} = 4/5$, $\Delta_{13} = 1/10$ and that vertex 1 is interdicted, that is, $x = (1, 0, 0, 0)'$. The probabilities for the arcs leaving vertex 1 including the self-loop change to

$$P'_{12} = \frac{1}{3} \cdot \frac{1}{5} = \frac{1}{15}, P'_{13} = \frac{1}{3} \cdot \frac{9}{10} = \frac{3}{10}, P'_{11} = \frac{1}{3} + \frac{1}{3} \cdot \frac{4}{5} + \frac{1}{3} \cdot \frac{1}{10} = \frac{19}{30}.$$

Replacing these probabilities in the equations results in:

$$\begin{aligned}
t_{13}(x) &= 1 + \frac{19}{30}t_{13}(x) + \frac{1}{15}t_{23}(x)\\
t_{23}(x) &= 1 + \frac{1}{3}t_{23}(x) + \frac{1}{3}t_{13}(x) + \frac{1}{3}t_{43}(x)\\
t_{43}(x) &= 1 + \frac{1}{3}t_{43}(x) + \frac{1}{3}t_{23}(x),
\end{aligned}$$

which gives that $t_{13}(x) = 3.72$, $t_{23}(x) = 5.48$, and $t_{43}(x) = 4.24$. Therefore, $t_{13}(x) < t_{13}$, and the FPT from between S and T in this case has decreased after interdiction according to x.

Intuitively, in the example in Figure 2, interdiction of vertex 1 makes it more likely for the DTMC to remain in vertex 1 (increasing the probability from 1/3 to 19/30) while at the same time decreasing the probability of transitioning from 1 to 3 only by small amount (from 1/3 to 3/10). This imbalance is because Δ_{13} is much smaller compared to Δ_{12}. Thus, one can expect interdiction to be successful, if for any $i \in V$, all the Δ_{ij} have similar values across all the outbound arcs. This observation is made precise in Theorem 1. Subsequently, in Theorem 2, we show that FPT interdiction is NP-hard even when restricted to such instances.

Theorem 1. *If the interdiction penalties Δ_{ij} on the outbound arcs depend only on the departing state, that is, for every $i \in V$, we have $\Delta_{ij} = \Delta_i \in [0, 1)$ for all $j \in N^+(i)$, then the FPTs are monotonically non-decreasing functions of the interdiction policies (partially) ordered by vertex inclusion. That is, for distinct $i \in V$ and $j \in T$, $t_{ij}(\bar{x}) \geq t_{ij}(\tilde{x})$ for $\bar{x}, \tilde{x} \in \{0, 1\}^{|V|}$ such that $\bar{x} \geq \tilde{x}$.*

Proof. For an arbitrary fixed vertex $j \in T$, after enforcing $t_{jj}(x) = 0$ and

$P_{ii} + \sum_{k \in N^+(i)} P_{ik} = 1$, we can rewrite equations (3) as follows:

$$t_{ij}(x) - \frac{1}{(1 - P_{ii})} \sum_{k \in N^+(i) \setminus \{j\}} P_{ik} t_{kj}(x) = \frac{1}{(1 - \Delta_i x_i)(1 - P_{ii})} \quad \forall i \in V \setminus \{j\}.$$

For given x and j, we express the foregoing system of $|V| - 1$ equations in as many unknown FPTs in matrix notation to facilitate our arguments. Let the unknown FPTs be denoted by the $(|V| - 1)$-dimensional column vector:

$$T(x) \coloneqq [t_{1j}(x), t_{2j}(x), \cdots, t_{j-1,j}(x), t_{j+1,j}(x), \cdots, t_{|V|,j}(x)]^\top,$$

and let Q denote the $(|V| - 1) \times (|V| - 1)$ matrix whose entries are defined below (with rows and columns indexed by $V \setminus \{j\}$ for convenience):

$$Q_{ik} \coloneqq \begin{cases} \frac{P_{ik}}{1 - P_{ii}}, & \text{for } k \in N^+(i) \setminus \{j\}, \\ 0, & \text{otherwise}, \end{cases} \qquad \forall i \in V \setminus \{j\},$$

and let I denote the $(|V| - 1) \times (|V| - 1)$ identity matrix. Note that Q is well-defined because, from our assumptions, $P_{ii} = 1$ if and only if $i = j$ and $T = \{i\}$. Furthermore, let $R(x)$ denote the right-hand side column vector with its i-th element defined as:

$$R(x)_i \coloneqq \frac{1}{(1 - \Delta_i x_i)(1 - P_{ii})} \qquad \forall i \in V \setminus \{j\}.$$

Then, the equations of FPTs in the new notations are as follows:

$$(I - Q)T(x) = R(x).$$

Since V is finite, the rows of Q sum to at most one, and at least one row of Q sums to strictly less than one (otherwise, the probability of reaching state j will be zero), then the absolute value of all eigenvalues of Q is strictly less than one (see e.g., Theorem 2.8 of [14]). Thus, $(I-Q)^{-1}$ exists and is given by the following geometric series expansion:

$$(I-Q)^{-1} = \sum_{\ell=0}^{\infty} Q^{\ell}.$$

Consequently,

$$T(x) = \sum_{\ell=0}^{\infty} Q^{\ell} R(x).$$

For i such that $\bar{x}_i > \tilde{x}_i$, we see that $R(\bar{x})_i > R(\tilde{x})_i$; with the components being equal otherwise. As each element in Q^{ℓ} is non-negative for all $\ell \geq 0$ and $R(\bar{x}) \geq R(\tilde{x})$, we conclude that $T(\bar{x}) \geq T(\tilde{x})$, establishing our claim. □

Corollary 1. *If the assumptions of Theorem 1 hold, then for any interdiction decision x, we can compute lower and upper bounds on FPTs based on the inequality: $t_{ij}(\vec{0}) \leq t_{ij}(x) \leq t_{ij}(\vec{1})$, where $\vec{0}$ and $\vec{1}$ are the $|V|$-dimensional all-zero and all-one vectors, respectively.*

3.2. FPT interdiction problem is NP-hard

In this section, we show that problem (2) is NP-hard using a polynomial-time reduction from vertex cover [15]. Consider the following decision problems as given below. We assume that all data are rational.

Vertex Cover: Given a simple, undirected graph $G = (V, E)$ and a positive integer b, does G contain a vertex cover of size at most b? In other words, is there a subset C with at most b vertices that contains at least one end point of every edge in G?

FPT Interdiction: Given a digraph $D = (V, A)$, a DTMC $\{X_n : n \geq 0\}$ with state space V and one-step transition probability matrix P, interdiction penalties $\Delta_{ij} \in [0, 1)$ for $i \in V, j \in N^+(i)$, disjoint nonempty vertex subsets S, T, a positive integer b, and a positive rational number ℓ, does there exist a feasible interdiction policy $x \in B$ such that $t_{ij}(x) \geq \ell$ for all $i \in S$ and $j \in T$?

Theorem 2. FPT INTERDICTION *is NP-complete.*

Proof. Given an interdiction policy x, in polynomial time we can verify its feasibility, compute the minimum FPT from S to T using the system of linear equations (3), and verify whether or not it is at least ℓ. Therefore, FPT INTERDICTION belongs to class NP.

We demonstrate a polynomial time reduction from VERTEX COVER to establish NP-hardness. Let $\langle G = (V, E), b\rangle$ denote the VERTEX COVER instance. Without loss of generality, we assume that G contains no isolated vertices. We create the digraph by adding self-loops at every vertex of G and replacing every edge of G with anti-parallel arcs. We also create a new auxiliary vertex $w \notin V$ to which every vertex in G has outbound arcs (see Figure 3). Denote this digraph by $D = (V \cup \{w\}, A)$, where,

$$A \coloneqq \bigcup_{i \in V \cup \{w\}} \{(i, w), (i, i)\} \cup \bigcup_{\{i,j\} \in E} \{(i, j), (j, i)\} .$$

We set $S = V$, $T = \{w\}$, the interdiction budget is b (same as the upper-bound of the vertex cover instance), and $\ell = 2$. We also use a parameter $p \in [0, 1)$ in defining the transition probabilities and interdiction penalties below, and eventually demonstrate the claim for a sufficiently large p. The one-step transition probability matrix of the DTMC is defined as follows, for each $(i,j) \in A$,

$$P_{ij} = \begin{cases} 0, & \text{if } i = j \neq w, \\ 1, & \text{if } i = j = w, \\ p, & \text{if } j = w \neq i, \\ (1-p)/d_i, & \text{otherwise,} \end{cases}$$

where, $d_i \coloneqq |N(i)|$ is the degree of vertex i and $N(i)$ denotes the neighborhood of vertex i in the vertex cover instance G. Note that the minimum degree of a vertex in G is at least one. Finally, assume that the interdiction penalties are $\Delta_{ij} = p$ for every $i \in V$ and $j \in N^+(i)$. This completely specifies the FPT Interdiction instance which can be constructed in polynomial time. Note that the constructed interdiction instance also satisfies the conditions of Theorem 1.

The FPTs for the interdiction instance is given by the following equations for any interdiction policy $x \in B$ and $i \in V$ using equation (3) and substituting Δ_i as p or zero as applicable:

$$t_{iw}(x) = \frac{1}{(1-px_i)} + \frac{(1-p)}{d_i} \sum_{k \in N(i)} t_{kw}(x). \tag{4}$$

Next we show that the FPT Interdiction instance is a 'yes' instance if and only if G contains a vertex cover of size at most b. Suppose $C \subset V$ is a vertex cover of G containing at most b vertices. Consider the interdiction policy x in which we interdict vertices in C. We claim that the FPT from an arbitrary vertex $i \in S$ to w is at least 2. We consider two cases: vertex $i \in C$ and vertex $i \notin C$.

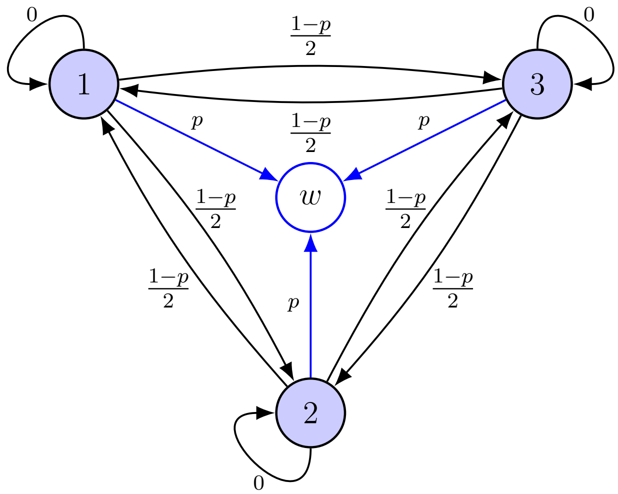

Figure 3: The digraph obtained by applying our transformation to a complete graph on three vertices. The self-loop of w with $P_{ww} = 1$ is not shown to simplify the picture.

Case (i) Vertex $i \in C$: In this case, $x_i = 1$ and the value of the first passage time is:

$$t_{iw}(x) = \frac{1}{(1-p)} + \frac{(1-p)}{d_i} \sum_{k \in N(i)} t_{kw}(x)$$

In this case, if we choose $p \geq 1/2$, then $t_{iw}(x) \geq 2$.

Case (ii) Vertex $i \notin C$: In this case, $x_i = 0$ and $x_k = 1$ for all $k \in N(i)$ as $N(i) \subset C$ to cover all the edges incident at vertex i. The FPT in this case is:

$$\begin{aligned} t_{iw}(x) &= 1 + \frac{(1-p)}{d_i} \sum_{k \in N(i)} t_{kw}(x) \\ &= 1 + \frac{(1-p)}{d_i} \sum_{k \in N(i)} \left[\frac{1}{(1-p)} + \frac{(1-p)}{d_k} \sum_{j \in N(k)} t_{jw}(x) \right] \end{aligned}$$

$$= 2 + \frac{(1-p)^2}{d_i} \sum_{k \in N(i)} \frac{1}{d_k} \sum_{j \in N(k)} t_{jw}(x).$$

Thus, $t_{iw}(x) \geq 2$ for every $i \in S$, as long as the constant p in the reduction is chosen so that $p \geq 1/2$. The interdiction instance is a 'yes' instance.

Now, assume that there is no size-b vertex cover in G. Therefore, for any interdiction set C of size at most b, there exists an arc $(i, j) \in A$ where neither of the vertices $i, j \in V$ are interdicted. Given these particular vertices i and j, by denoting by $\Pr_i[\cdot] \coloneqq \Pr[\cdot | X_0 = i]$ (and analogously for expectations), the expected value $t_{iw}(x)$ can be equivalently calculated as follows:

$$t_{iw}(x) = \Pr_i[\tau_w = 1] + 2 \Pr_i[\tau_w = 2] + \mathrm{E}_i[\tau_w | \tau_w \geq 3] \Pr_i[\tau_w \geq 3]. \tag{5}$$

Note that $\Pr_i[\tau_w = 1] = p$ and

$$\begin{aligned}
\Pr_i[\tau_w = 2] &= \sum_{k \in N(i) \setminus C} P_{ik} P_{kw} + \sum_{k \in N(i) \cap C} P_{ik} P_{kw} \\
&= \frac{(1-p)}{d_i} \sum_{k \in N(i) \setminus C} p + \frac{(1-p)}{d_i} \sum_{k \in N(i) \cap C} p(1-p) \\
&= \frac{p(1-p)}{d_i} \big(d_i^- + (1-p) d_i^+ \big),
\end{aligned}$$

where $d_i^- \coloneqq |N(i) \setminus C|$ and $d_i^+ \coloneqq |N(i) \cap C|$. Note that $d_i = d_i^+ + d_i^-$ and $d_i \geq d_i^- \geq 1$ as $j \in N(i) \setminus C$. Moreover,

$$\begin{aligned}
\Pr_i[\tau_w \geq 3] &= 1 - \Pr_i[\tau_w = 1] - \Pr_i[\tau_w = 2] \\
&= (1-p) \left(1 - \frac{p(d_i^- + (1-p) d_i^+)}{d_i} \right), \text{ and}
\end{aligned}$$

it can be verified using (4) that $t_{iw}(x) \leq t_{iw}(\vec{1}) = 1/(p(1-p))$. By exploiting the Markovian property, this observation implies that:

$$\mathrm{E}_i[\tau_w | \tau_w \geq 3] \leq 3 + \frac{1}{p(1-p)}.$$

Therefore, from (5) we can conclude that:

$$t_{iw}(x) \leq p + \frac{1}{p} - \frac{[p(1-p)+1][d_i^- + (1-p)d_i^+]}{d_i} + 3(1-p) \tag{6a}$$

$$\leq 3 - 2p + \frac{1}{p} - \frac{p(1-p)}{|V|} - \frac{1}{|V|}, \tag{6b}$$

where the last inequality follows as $d_i^-/d_i \geq 1/|V|$ and $d_i^+/d_i \geq 0$. It can be checked that if $p = 1 - 1/|V|^2$ then the right hand of (6b) is strictly less than 2 as long as $|V| \geq 3$. We have established that if G has no vertex cover of size at most b, i.e., a 'no' instance, then, there exist at least one $i \in S$ with $t_{iw}(x) < 2$ for any interdiction solution x that removes at most b vertices. Therefore, the interdiction instance is also a 'no' instance and the result follows. □

4. An MILP formulation

In this section we present a mixed-integer linear programming (MILP) formulation of problem (2). As before, we let a vector of binary decision variables $x \in \{0,1\}^{|V|}$ denote the interdiction set $\{i \in V : x_i = 1\}$. For convenience, we denote by t_{ij} the expected first passage time from i to j after interdiction, i.e., $t_{ij} \equiv t_{ij}(x)$, and by θ the smallest FPT from S to T. The optimization problem (2) can be formulated as:

$$t^*_{S,T} = \max \theta \tag{7a}$$

$$\sum_{i \in V} x_i \leq b \tag{7b}$$

$$\theta \leq t_{ij} \qquad \forall (i,j) \in S \times T \tag{7c}$$

$$\Big(1 - P_{ii} - \sum_{k \in N^+(i)} P_{ik}\Delta_{ik}x_i\Big)t_{ij} =$$

$$1 + \sum_{k \in N^+(i)} P_{ik}(1 - \Delta_{ik} x_i) t_{kj} \qquad \forall (i,j) \in V \times T : i \neq j \tag{7d}$$

$$t_{jj} = 0 \qquad \forall j \in V \tag{7e}$$

$$t_{ij} \geq 0 \qquad \forall (i,j) \in V \times V \tag{7f}$$

$$x_i \in \{0, 1\} \qquad \forall i \in V \tag{7g}$$

Constraint (7c) requires θ to be smaller than any first passage time in the digraph D from S to T. Constraints (7d) and (7e) are essentially equations (3), except those governing t_{ij} for $j \in V \setminus T$ are excluded as they do not impact FPTs from S to T. This formulation is not linear because of the variable products in constraint (7d). We introduce variables z_{ij} and y_{ij} to effectively replace constraint (7d) with the following equations:

$$(1 - P_{ii}) t_{ij} = 1 - z_{ij} + \sum_{k \in N^+(i) \setminus \{j\}} P_{ik} t_{kj}$$

$$z_{ij} = y_{ij} x_i$$

$$y_{ij} = \sum_{k \in N^+(i)} P_{ik} \Delta_{ik} t_{kj} - \sum_{k \in N^+(i)} P_{ik} \Delta_{ik} t_{ij}$$

The nonlinear equation $z_{ij} = y_{ij} x_i$ is enforced using 'big-M' coefficients on x_i and $1 - x_i$ in the MILP formulation that follows.

$$t^*_{S,T} = \max \theta \tag{8a}$$

$$\sum_{i \in V} x_i \leq b \tag{8b}$$

$$\theta \leq t_{ij} \qquad \forall (i,j) \in S \times T \tag{8c}$$

$$(1 - P_{ii}) t_{ij} = 1 - z_{ij} + o \sum_{k \in N^+(i)} P_{ik} t_{kj} \qquad \forall (i,j) \in V \times T : i \neq j \tag{8d}$$

$$y_{ij} = \sum_{k \in N^+(i)} P_{ik} \Delta_{ik} t_{kj} - \sum_{k \in N^+(i)} P_{ik} \Delta_{ik} t_{ij} \quad \forall (i,j) \in V \times T : i \neq j \tag{8e}$$

$$-M_{ij}(1-x_i) \le z_{ij} - y_{ij} \le M_{ij}(1-x_i) \quad \forall (i,j) \in V \times T : i \ne j \tag{8f}$$

$$-M_{ij}x_i \le z_{ij} \le M_{ij}x_i \quad \forall (i,j) \in V \times T : i \ne j \tag{8g}$$

$$t_{jj} = 0 \quad \forall j \in V \tag{8h}$$

$$t_{ij} \ge 0 \quad \forall (i,j) \in V \times V \tag{8i}$$

$$x_i \in \{0,1\} \quad \forall i \in V. \tag{8j}$$

To model the correct behavior of constraints (8f) and (8g), for a given $i, j \in V$, we need M_{ij} to satisfy the following inequality:

$$M_{ij} \ge \sum_{k \in N^+(i)} P_{ik}\Delta_{ik}t_{kj} - \sum_{k \in N^+(i)} P_{ik}\Delta_{ik}t_{ij},$$

for any feasible solution to formulation (8). For instance, under the assumptions of Theorem 1, using Corollary 1 we can compute a valid M_{ij} as:

$$M_{ij} = \sum_{k \in N^+(i)} P_{ik}\Delta_i t_{kj}(\vec{1}) - \sum_{k \in N^+(i)} P_{ik}\Delta_i t_{ij}(\vec{0}).$$

5. Numerical illustration

In this section, we present the results of implementing formulation (8) in C++, compiled using Microsoft® Visual Studio® 2017, and solved using Gurobi™ Optimizer v9.5.2 [16]. The purpose of these experiments is primarily illustrative, to demonstrate the extent by which FPTs are increased by interdiction and to show that small- to medium-sized instances can be solved by the MILP formulation with commercial solvers. Selected graphs from the Tenth DIMACS Implementation Challenge, which includes some popular social and biological network benchmarks, were considered in our test bed. We also included 20 instances used in [17], those we refer to as

Club instances, which are challenging benchmarks for the maximum k-club problem. All the graphs in this test bed are undirected, and we convert them to directed graphs by adding self-loops and replacing edges by anti-parallel arcs. Experiments are conducted on a 64-bit Windows® 10 Pro machine with 16GB of RAM and a 1.8 GHz processor with 7 cores.

In our experiments, sets S and T are chosen uniformly at random and their cardinality is equal to 20% of the number of vertices. The interdiction budget b is also equal to 20% of the number of vertices. For every vertex $i \in V$, transition probabilities of its outbound arcs and the self-loop are equal to $1/|N^+[i]|$. The interdiction penalties are set as $\Delta_{ij} = 0.5$ for every arc $(i, j) \in A$ such that $i \neq j$.

We report the results for the DIMACS-10 instances in Table 1. All the instances in this test bed are solved to optimality at the root node of the branch-and-bound tree. The percentage increase in the smallest first passage time as a result of interdiction ranges from 23% (for `football`) and to 55% (for `celegansm`). On this test bed we observe that an optimal interdiction policy delays the first passage times by significant amounts.

However, by increasing the number of vertices in the graphs, the number of variables and constraints of formulation (8) increase quadratically. So for larger networks (e.g., tens of thousands of arcs), the solver encounters memory crashes even while building the model. Therefore, solving the FPT interdiction problem for large scale instances will require more specialized decomposition methods.

The results for the Club instances are reported in Table 2. The main difference with respect to the DIMACS-10 instances is that there are six

Table 1: Results of solving formulation (8) using Gurobi. Minimum FPT before and after interdiction, along with the corresponding percentage increase are reported under column headings FPT before, FPT after, and % increase, respectively. Wall-clock running time is reported under column heading Time (in seconds).

Graph	$\|V\|$	$\|A\|$	FPT before	FPT after	% increase	Time (s)
karate	34	78	56.02	79.56	42.03	0.20
dolphins	62	159	38.77	52.60	35.66	0.38
lesmis	77	254	56.91	81.75	43.66	0.54
polbooks	105	441	54.87	77.67	41.57	1.16
adjnoun	112	425	39.69	57.13	43.96	1.26
football	115	613	100.94	124.38	23.22	2.16
jazz	198	2742	70.85	98.22	38.62	12.88
celegansn	297	2148	231.97	331.90	43.08	31.03
celegansm	453	2025	30.50	47.48	55.66	44.03
email	1133	5451	1011.17	1506.52	48.98	1755.68
add20	2395	7462	3994.62	6433.78	61.06	696.43

instances that are not solved at the root node and there are 11 instances where either Gomory or RLT cuts are added by the solver. Although there are several Club instances that utilized general purpose cutting planes to arrive at an integral optimum (either at the root node or after enumeration), it is notable that for 9 Club instances and all DIMACS-10 instances the optimal solution of formulation (8) is obtained at the root node without the addition of any cutting planes. Our preliminary analysis did not uncover any simple explanation for this behavior. It would be interesting to see if graph properties like edge sparsity, low graph degeneracy, or small treewidth of the underlying graph make the problem easier to solve.

Table 2: Results of solving formulation (8) using Gurobi. Minimum FPT before and after interdiction, along with the corresponding percentage increase are reported under column headings FPT before, FPT after, and % increase, respectively. Wall-clock running time is reported under column heading Time (in seconds). Columns '#BC nodes' and '#Cuts' respectively report the number of branch-and-cut nodes enumerated and the number of Gomory cuts added. For instances `graph_200_0.15-5` and `graph_200_0.15-6`, in addition to the Gomory cuts, one RLT cut is also added.

Graph	$\|V\|$	$\|A\|$	FPT before	FPT after	% increase	Time (s)	#BC nodes	#Cuts
graph_200_0.1(1)	200	2015	130.15	164.14	26.12	42.85	1	2
graph_200_0.1(2)	200	1983	140.63	177.29	26.07	58.51	15	2
graph_200_0.1(3)	200	2014	149.08	187.88	26.03	15.04	1	0
graph_200_0.1(4)	200	1956	150.37	189.64	26.12	43.84	1	1
graph_200_0.1(5)	200	2033	114.98	145.45	26.50	15.29	1	0
graph_200_0.1(6)	200	1971	137.71	174.28	26.55	13.19	1	0
graph_200_0.1(7)	200	2029	150.14	189.01	25.89	34.26	1	1
graph_200_0.1(8)	200	2037	135.47	171.27	26.42	139.45	655	3
graph_200_0.1(9)	200	2009	163.60	205.11	25.37	41.33	1	0
graph_200_0.1(10)	200	1999	149.06	189.42	27.08	15.57	1	0
graph_200_0.15(1)	200	2981	143.99	179.77	24.85	25.01	1	0
graph_200_0.15(2)	200	3028	167.13	207.89	24.39	392.63	1131	7
graph_200_0.15(3)	200	2964	149.60	187.78	25.52	23.46	1	0
graph_200_0.15(4)	200	3035	147.04	184.58	25.53	64.55	1	1
graph_200_0.15(5)	200	2887	134.24	167.24	24.58	65.69	1	6
graph_200_0.15(6)	200	3065	153.08	190.58	24.49	83.31	23	1
graph_200_0.15(7)	200	2961	150.08	187.47	24.91	23.88	1	0
graph_200_0.15(8)	200	3055	138.09	172.95	25.25	72.32	3	1
graph_200_0.15(9)	200	2987	143.81	179.50	24.81	49.40	1	0
graph_200_0.15(10)	200	3077	148.62	184.58	24.19	112.94	239	1

6. Conclusions

We introduced an interdiction model to maximize the minimum first passage time between two given sets of states in a DTMC. We demonstrate that the interdiction strategy could be counter-productive in some situations and provide a sufficient condition that guarantees that interdiction will not decrease the first passage times. We then established the NP-hardness of the problem. We present a MILP formulation that can be used to solve small to medium sized instances of the problem using a commercial solver within minutes. We observed from our preliminary experiments that the interdiction framework we propose is capable of producing significant increases in the minimum FPT, demonstrating its potential applicability. It would be interesting in light of some of our computational results to identify graph classes for which this problem may be polynomially solvable. More generalized interdiction budget constraints of the form $B \coloneqq \left\{ x \in \{0,1\}^{|V|} : \sum_{(i,j)\in A} c_{ij} \Delta_{ij} x_i \leq b \right\}$ could also be explored in specific applications. In order to ensure that the methodology can scale well in practice, future work needs to focus on decomposition techniques to solve our formulation and also explore alternate formulation ideas.

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